Chapter 6: Inductance, Capacitance and Mutual Inductance

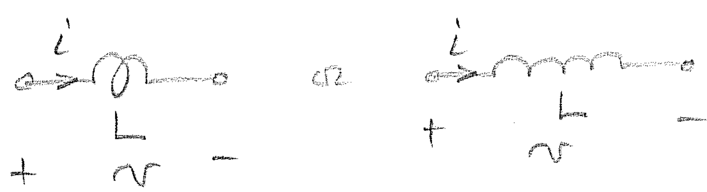
A resistance dissipates energy. It does not store it. An inductance stores energy in the form of a magnetic field, whereas a capacitance stores energy in the form of an electric field.

Inductances and capacitances cannot create energy they can store energy and energy stored can be extracted. In that respect they are passive elements.

Capacitance are used in capacitive proximity switches. When one touches such a switch, a change in the value of the capacitance occurs causing a voltage change which activates the switch.

6.1 The Inductor

Inductance is the circuit parameter used to describe an inductor. Its symbol is L and is represented by a coiled wire:



Note that L is measured in henrys (H).

The voltage across an inductor is proportional to the rate of change of current:

$$v = L \frac{di}{dt}$$

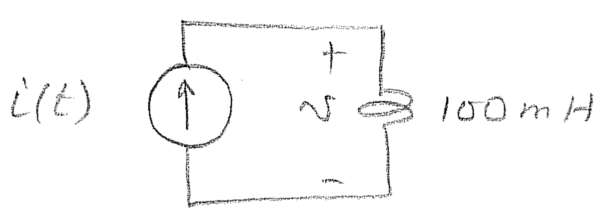
So when the current is constant, the voltage across the inductor is zero! Thus the inductor behaves as a short circuit. Also, the current across an inductor cannot change instantaneously as this would require infinite voltage.

Attempts to open a circuit on a large inductive current creates arcing and high voltages in the circuit that lead to damaging circuit components

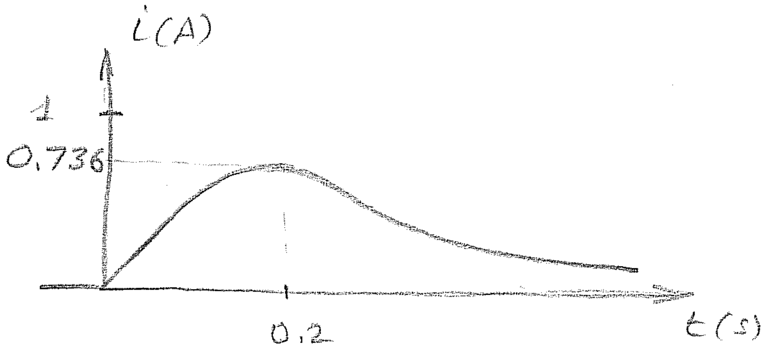
Example 6.1:

The current in the circuit is given by:

$$i(t) = \begin{cases} 0 & t \leq 0 \\ 10te^{-5t}\ A & t > 0 \end{cases}$$



a) Draw $i(t)$ and determine t_{max} when $i(t)$ is maximum.



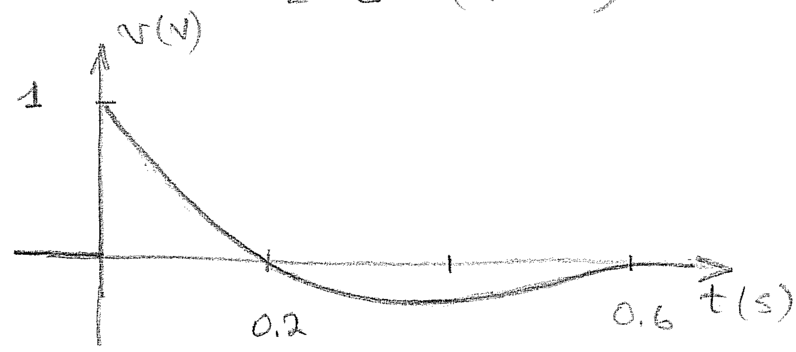
$i(t)$ is maximum when $\frac{di}{dt} = 0$!

$$\frac{di}{dt} = 10\left(e^{-5t} - 5te^{-5t}\right) = 0 \Rightarrow t = \frac{1}{5} = 0.2\,s.$$

$$i\,(t=0.2) = 10 \times 0.2\, e^{-5\times 0.2} = 0.736\ A.$$

b) Determine the voltage and sketch its waveform.

$$v = L\frac{di}{dt} = 100\times 10^{-3} \times 10\, e^{-5t}(1-5t)\ \ V$$
$$= e^{-5t}(1-5t)\,V$$



Note the voltage has a zero value at the maximum of the current. At this instant there is a change of polarity.

At $t=0$ there is an instantaneous change in the voltage, but there is no instantaneous change in current.

Current in L in terms of v:

By integrating equation (6.1), the voltage of an inductor can be expressed in terms of its current:

$$v = L\frac{di}{dt} \Rightarrow v\,dt = L\,di \qquad (6.3)$$

By integrating:

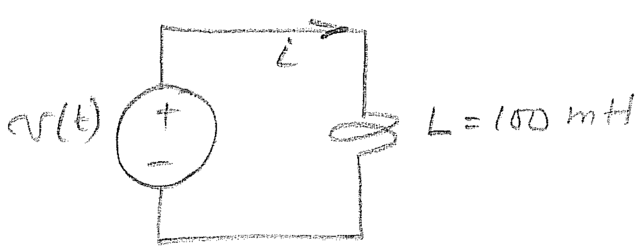
$$\int_{t_0}^{t} v\,d\tau = L\int_{i(t_0)}^{i(t)} dk$$

we used τ and k as variables of integration:

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(\tau)\,d\tau + i(t_0)$$

Example 6.2:

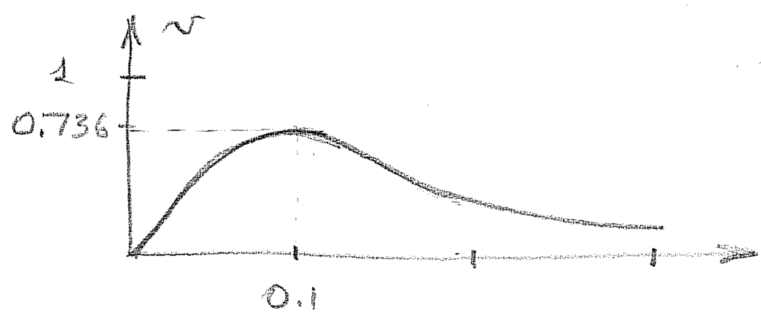
A voltage pulse is applied to the 100 mH inductor in the following circuit:



$v(t) = 20te^{-10t}$ V

The current i for $t \leq 0$ is zero.

a) Sketch the voltage as a function of time.



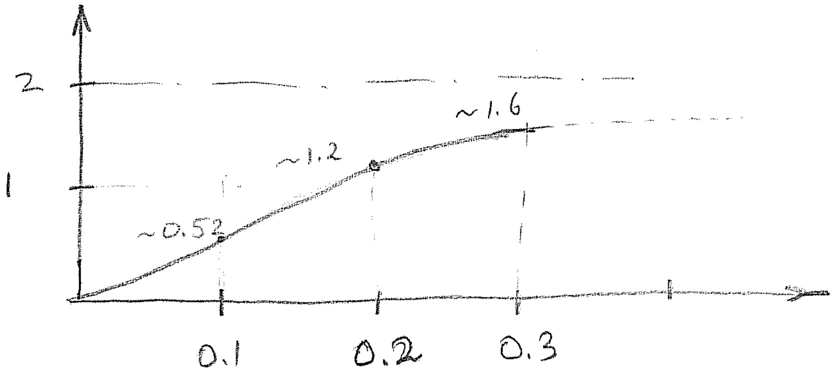
b) Find the inductor as a function of time

$$i(t) = \frac{1}{0.1}\int_0^t 20\tau e^{-10\tau}\, d\tau + 0$$

$$= 200\left(\frac{-e^{-10\tau}}{100}(10\tau+1)\right)\Big|_0^t = 2\left(1 - 10te^{-10t} - e^{-10t}\right) A$$

for $t > 0$

The sketch of the current is shown below:



As $t \to \infty$ $i(t) \to 2$ A

Power and Energy in the Inductor.

Power in general is given by $p = vi$, for an inductor $v = L\frac{di}{dt}$, so:

$$p = Li\frac{di}{dt} \quad W$$

We can also express the power in terms of the voltage:

$$p = v\left(\frac{1}{L}\int_{t_0}^{t} v\,d\tau + i(t_0)\right)$$

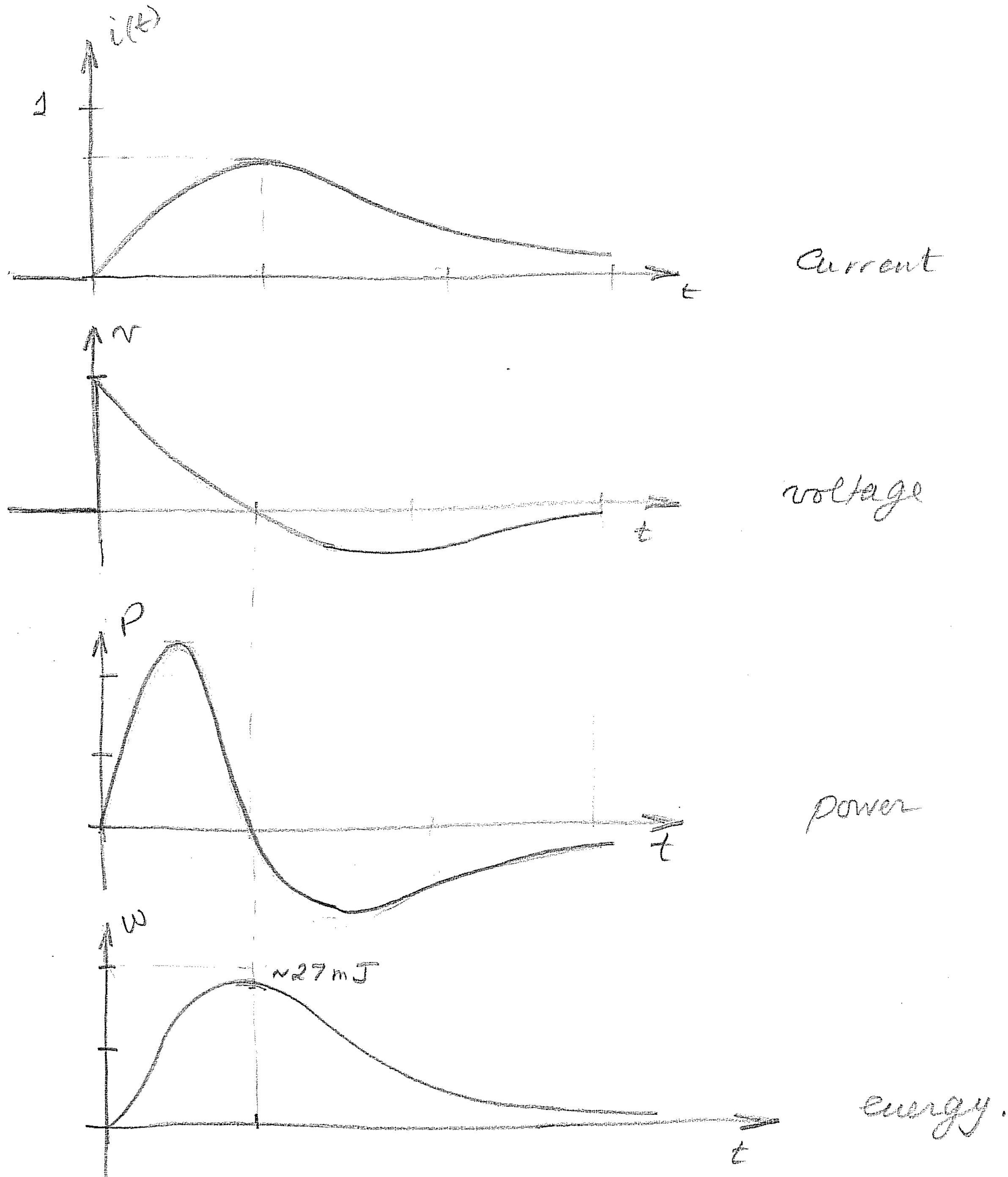
The energy stored in an inductor can be obtained by integrating the power:

$$w = \int_{t_0}^{t} p(\tau)\,d\tau = L\int_{i(t_0)}^{i(t)} i\,di$$

$$w(t) = \frac{1}{2}Li^2(t).$$

Example 6.3

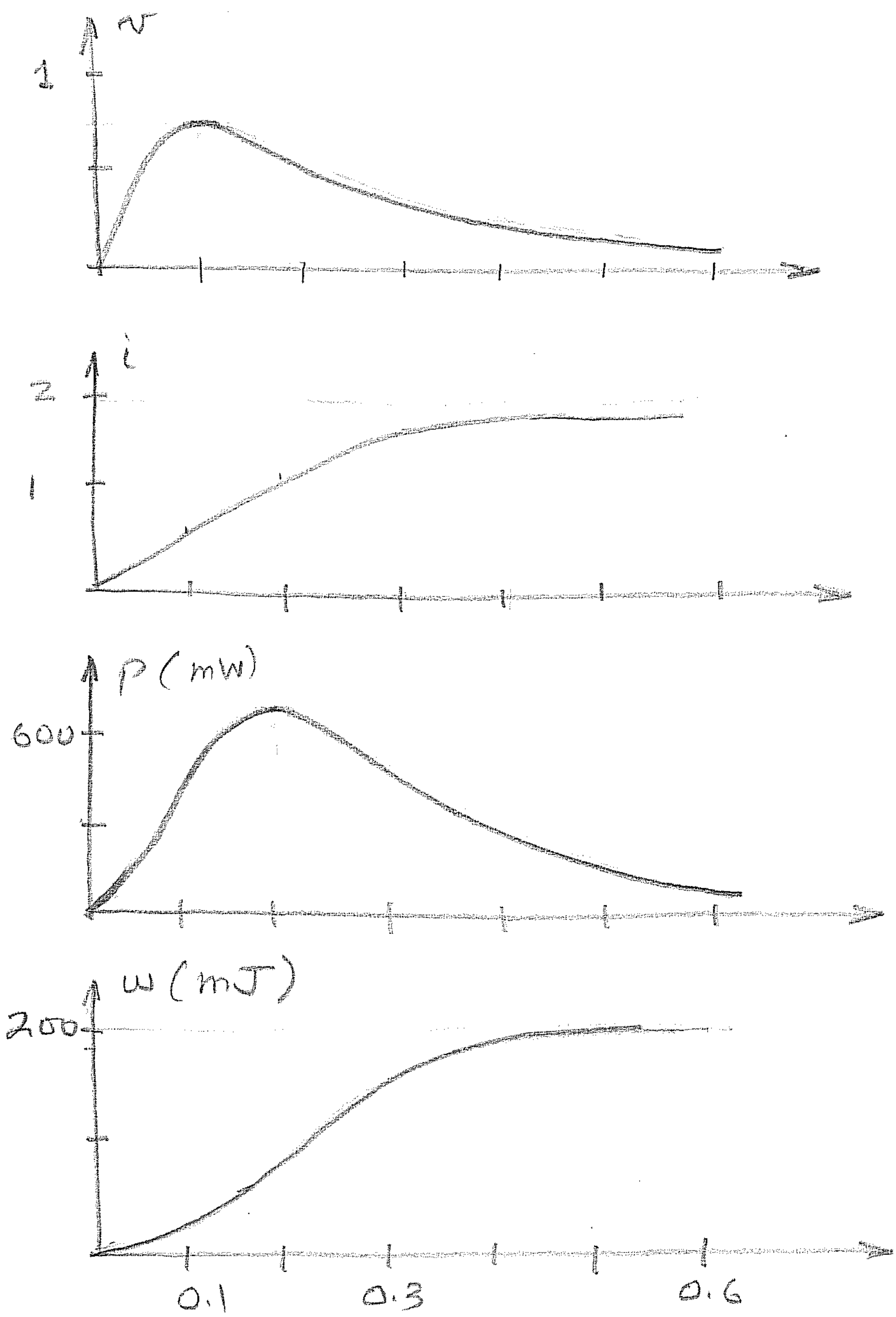
a) Plot i, v, p and w for example 6.1



energy is being stored in the inductor from $t=0$ to $t=0.2\,s$. Energy is extracted from the inductor from $t=0.2\,s$ to $t \to \infty$. The maximum energy stored can be calculated from $w=\frac{1}{2}Li^2$ at $t=0.2s$ which gives 27.07 mJ. When $P>0$ energy is being stored and when $P<0$ energy is being extracted.

The integral $w_1 = \int_0^{0.2} p\,dt$ is the energy stored in L at $t = 0.2\,s$ and $w_2 = \int_{0.2}^{\infty} p\,dt$ is the energy extracted from the inductor. The value of $w_1 = 27.07\,mJ$ and $w_2 = -27.07\,mJ$.

b) Plot v, i, p and w for example 6.2:



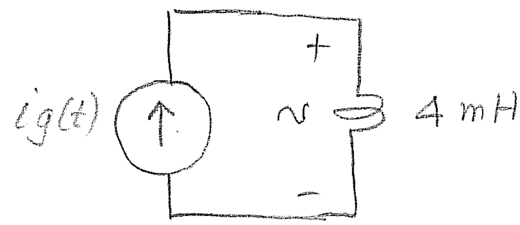
The energy in the inductor is not dissipated (no resistance). The energy approaches 200 mJ as $t \to \infty$ because the current approaches 2A as $t \to \infty$!

Assessment Problem 6.1

The current source in the circuit shown is given by:

$$i_g(t) = \begin{cases} 0 & t < 0 \\ 8\left(e^{-300t} - e^{-1200t}\right) & t \geq 0 \end{cases}$$

a) Find $v(0)$:



$$v(t) = L\frac{di}{dt}$$

$$= 8\left(-300e^{-300t} + 1200e^{-1200t}\right) \times 4 \times 10^{-3}$$

$$= 32\left(-0.3e^{-300t} + 1.2e^{-1200t}\right)$$

$$v(0) = 32\left(-0.3 + 1.2\right) = 28.8\,V$$

b) Find $t_1 > 0$ when $v(t_1) = 0$

$$v(t_1) = 0 \Rightarrow \quad 1.2e^{-1200t_1} - 0.3e^{-300t_1} = 0 \Rightarrow$$

$$\ln\left(\frac{1.2}{0.3}\right) = \ln\left(e^{1200t - 300t}\right)$$

$$= (900t_1) \Rightarrow$$

$$t_1 = \ln\left(\frac{1.2}{0.3}\right) \times \frac{1}{900} = 1.54 \times 10^{-3}\,s.$$

c) Power delivered to the inductor is given by:

$$p = vi = 8\left(e^{-300t} - e^{-1200t}\right) \times 32 \times \left(1.2e^{-1200t} - 0.3e^{-300t}\right)$$

$$\Rightarrow \quad p = 256\left(1.2e^{-1500t} - 0.3e^{-600t} - 1.2e^{-2400t} + 0.3e^{-1500t}\right)$$

$$\Rightarrow \quad p = 384e^{-1500t} - 76.8e^{-600t} - 307.2e^{-2400t}$$

d) The power is maximum when $\frac{dp}{dt} = 0$.

$$\frac{dp}{dt} = \underset{\alpha}{76.8 \times 600}\, e^{-600t} - \underset{\beta}{384 \times 1500}\, e^{-1500t} + \underset{\gamma}{307.2 \times 2400}\, e^{-2400t} = 0$$

So $\quad \gamma e^{-2400t} = \beta e^{-1500t} - \alpha e^{-600t}$

Multiply by e^{-600t} and take the natural log and rearrange to obtain:

$$t = \frac{\ln \gamma - \ln(\beta e^{-900t} - \alpha)}{1800}$$

The above equation defines an iterative process. Given an estimate t_i of the time of the peak, plug it into the rhs of the above equation to obtain an improved estimate t_2, and so on.

An initial estimate is obtained by the time at which the voltage becomes zero i.e. $t = 1.54$ ms. The power must be at a maximum prior to this time. As an initial estimate select $t_1 = \frac{1.54}{2} \simeq 0.75 \times 10^{-3}$. Replace in the rhs to obtain $t_2 = 0.606 \times 10^{-3}$. The process converges. At iteration 12 $\quad t_{12} = 0.4115$ ms.

$t_{max} = 0.4115$ ms

e) $p(t_{max}) = 32.72$ W

f) The energy stored is given by $w = \frac{1}{2} L i^2$, so it is maximum when i is maximum at $t_1 = 1.54$ ms

g) The current at $t_1 = 1.54$ ms is:

$$i(t_1) = 8\left(e^{-300 \times 1.54 \times 10^{-3}} - e^{-1200 \times 1.54 \times 10^{-3}}\right)$$

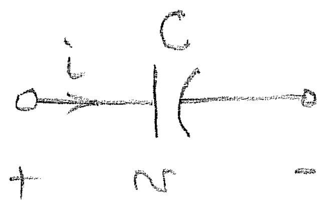
$$= 3.78 \text{ A}$$

$$w = \frac{1}{2} L i^2 = \frac{1}{2}\, 4 \times 10^{-3} \times 3.78^2 = 28.57 \text{ mJ.}$$

6.2 The Capacitor

A capacitor stores energy in the form of electric charges on its two plates which are seperated by a dielectric or insulating material. The symbol of a capacitance is C and it is measured in farads (F). As charge accumulates on the plates of a capacitor a current is observed and is known as a displacement current. Note no charge can move across the dielectric.

The graphic symbol and voltage-current polarities are shown as:



The current is proportional to the rate of change of voltage with respect to time:

$$i = C\frac{dv}{dt} \qquad (6.13)$$

Two observations: The voltage cannot change instantaneously as this would require infinite current. This is a physical impossibility. Second if the voltage across the terminals is constant, then the current is zero. Physically, when the voltage is constant, there is no charge accumulation at the capacitor's plates and thus its current is zero.

The capacitor voltage may be expressed in terms of the current by integrating equation 6.13:

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)\, d\tau + v(t_0).$$

The power delivered to a capacitor is given by:

$$p = v i = C v \frac{dv}{dt}$$

or in terms of the current:

$$p = i\left(\frac{1}{C}\int_{t_0}^{t} i\, d\tau + v(t_0)\right)$$

Since power p is the rate of change of energy we have:

$$p = \frac{dw}{dt} = C v \frac{dv}{dt} \Rightarrow$$

$$\int_0^{w} dx = C\int_0^{v} y\, dy \Rightarrow$$

$$w = \frac{1}{2} C v^2$$

Example 6.4: Current, voltage, power, and energy for a capacitor

The following voltage pulse is applied to the terminal of a capacitor with $C = 0.5\,\mu F$:

$$v(t) = \begin{cases} 0 & t \leq 0 \text{ s} \\ 4t \text{ V} & 0 \leq t < 1 \text{ s} \\ 4e^{-(t-1)} \text{ V} & t \geq 1 \text{ s} \end{cases}$$

a) Derive the expressions for the capacitor current, power and energy.

The current is determined from $i = C\frac{dv}{dt}$.

$i = 0$ for $t \leq 0$

$i = 0.5 \times 10^{-6} \times 4 = 2\,\mu A \qquad 0 < t < 1 \text{ s}$

$i = 0.5 \times 10^{-6} \times (-4e^{-(t-1)}) = -2e^{-(t-1)}\,\mu A \qquad t \geq 0$

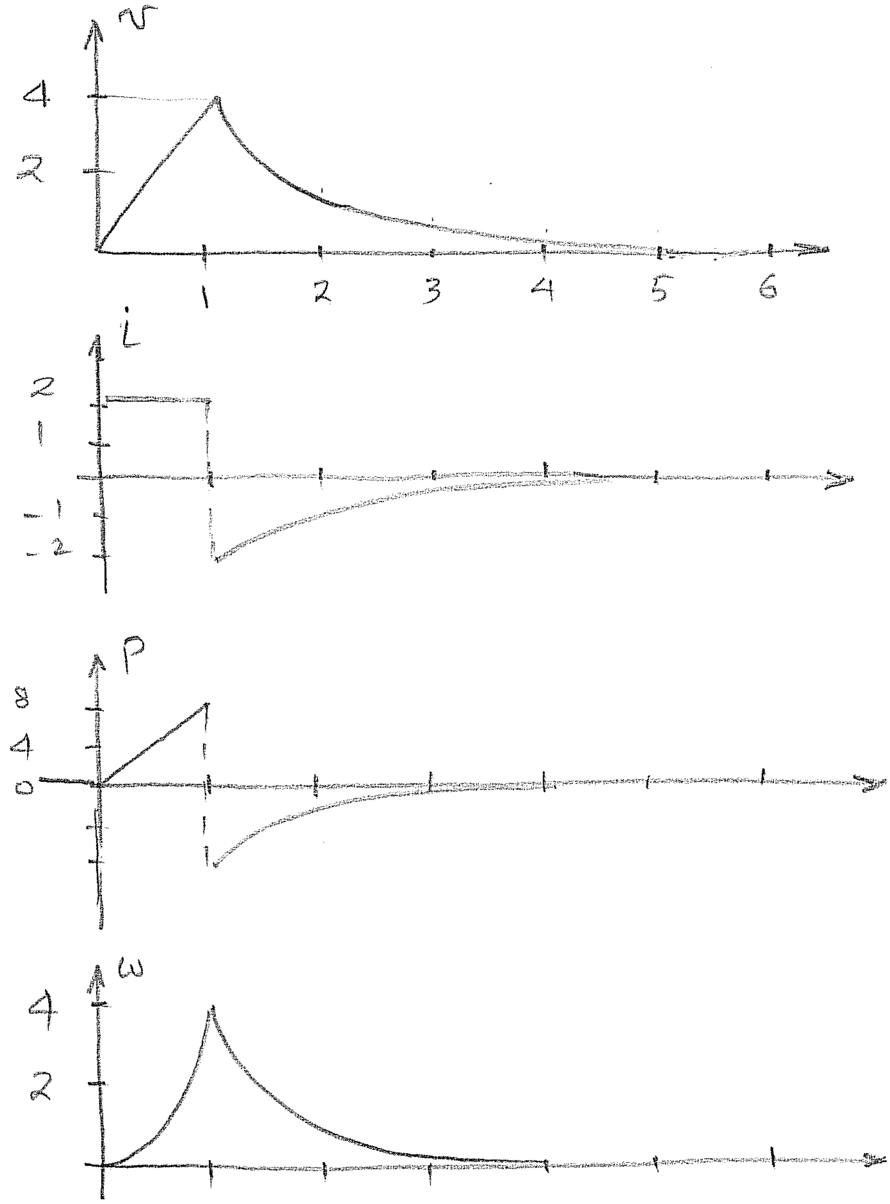
The power is given by: $p = vi$

$$p = \begin{cases} 0 & t \leq 0 \\ 8t\ \mu W & 0 < t < 1 \text{ s} \\ -8e^{-2(t-1)} & t \geq \text{ s} \end{cases}$$

The energy is given by $w = \frac{1}{2}Cv^2$

$$w = \begin{cases} 0 & t \leq 0 \text{ s} \\ \frac{1}{2}\, 0.5 \times 16t^2 = 4t^2\,\mu J & 0 < t < 1 \text{ s} \\ \frac{1}{2}\, 0.5 \times 16\, e^{-2(t-1)} = 4e^{-2(t-1)} & t \geq 1 \text{ s} \end{cases}$$

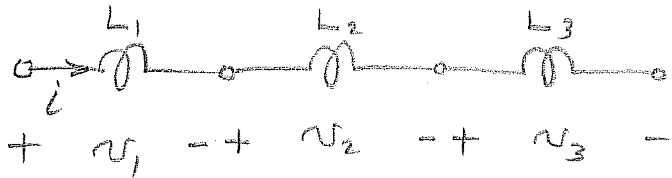
b) Sketch the voltage, current, power and energy.



When $p>0$ energy is being stored in the capacitor (v increases and w increases). When $p<0$ energy is being delivered by the capacitor (w is decreasing)
The area under the p-curve is the energy delivered to the capacitor when $p>0$ ($t \leq 1s$), and the area under the p-curve when $t \geq 1s$ is the energy delivered by the capacitor.

6.3 Series-Parallel Combinations of Inductance and Capacitance

<u>Series inductances</u> share the same current as illustrated:



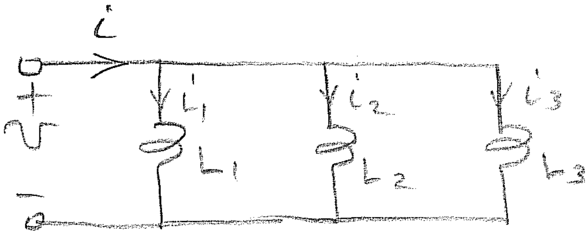
$$v_1 = L_1 \frac{di}{dt}, \quad v_2 = L_2 \frac{di}{dt}, \quad v_3 = L_3 \frac{di}{dt}$$

The voltage across the series combination is:

$$v = v_1 + v_2 + v_3 = (L_1 + L_2 + L_3)\frac{di}{dt}$$

So $\qquad L_{eq} = L_1 + L_2 + L_3$

Parallel inductances share the same voltage, as illustrated:



The currents are:

$$i_1 = \frac{1}{L_1}\int_0^t v\,d\tau + i_1(0)$$

$$i_2 = \frac{1}{L_2}\int_0^t v\,d\tau + i_2(0)$$

$$i_3 = \frac{1}{L_3}\int_0^t v\,d\tau + i_3(0)$$

The total current i is given by:

$$i = i_1 + i_2 + i_3 \Rightarrow$$

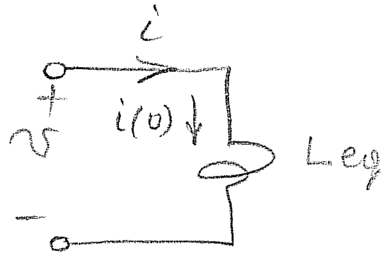
$$i = \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} \right) \int_0^t v \, d\tau + i_1(0) + i_2(0) + i_3(0)$$

The above equation may be interpreted in terms of a single conductor as:

$$i = \frac{1}{L_{eq}} \int_0^t v \, d\tau + i(0)$$

with

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} \quad \text{and} \quad i(0) = i_1(0) + i_2(0) + i_3(0).$$



Capacitors in series share the same current, thus they combine in a manner similar to inductors in parallel. So:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \qquad v(0) = v_1(0) + v_2(0) + v_3(0)$$

Capacitors in parallel share the same current; they combine as inductors in series. So

$$C_{eq} = C_1 + C_2 + C_3 \qquad v(0) = v_1(0) = v_2(0) = v_3(0).$$

6.4 Mutual Inductance

Inductance is the parameter that relates voltage to a time varying current in the same circuit; so inductance is also refered to as self-inductance.

When two coils are linked by a magnetic field, the voltage induced in one coil can be related to the time-varying current in the other using mutual inductance.

Consider two magnetically coupled coils shown below:

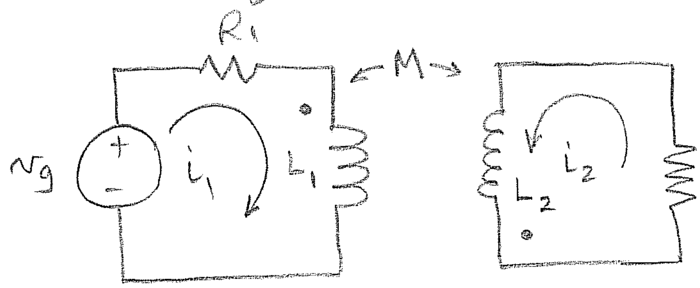


Fig. 6.21

In coil 1 there are two voltages one due to the self inductance, e.g. $L_1 \frac{di_1}{dt}$ and the second is $M\frac{di_2}{dt}$. The polarity of $M\frac{di_2}{dt}$ depends on the way the coils are wound. Showing the details of windings is cumbersome. Instead we use the <u>dot convention</u> where a dot is placed on one terminal of each winding. The rule for using the dot convention is as follows:

"When the current enters the dotted terminal of a coil, the voltage induced in the other coil is positive at its dotted terminal".

For the given dot markings in the above figure, current i_2 exits the dotted terminal, so the voltage it induces has a negative polarity at the dotted terminal of coil 1. So the voltage $M\frac{di_2}{dt}$ is a voltage rise in the direction of i_1.

So if we now write KVL for loop 1, we have:

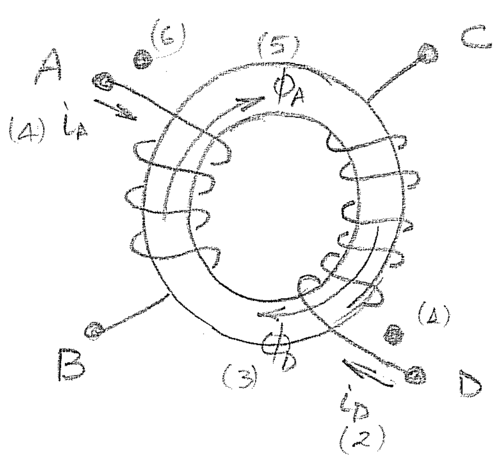
$$-v_g + Ri_1 + L_1\frac{di_1}{dt} - M\frac{di_2}{dt} = 0.$$

Current i_1 enters the dotted terminal of coil 1, so the voltage it induces in coil 2 is positive at the dotted terminal of coil 2. In the direction of the current it is also a voltage rise.

So KVL for loop 2 is:

$$R_2 i_2 + L_2\frac{di_2}{dt} - M\frac{di_1}{dt} = 0$$

Usually, the dot markings are given. We will now discuss two methods of determining dot markings. In the first the physical arrangement of the coils as shown below:



The following 6 steps are applied to determine the dot markings:

i) Select, arbitrarily one terminal, say D, and mark it with a dot;

ii) Assign a current into the dotted terminal i_D;

iii) Determine the direction of the magnetic field ϕ_D using the right-hand-rule;

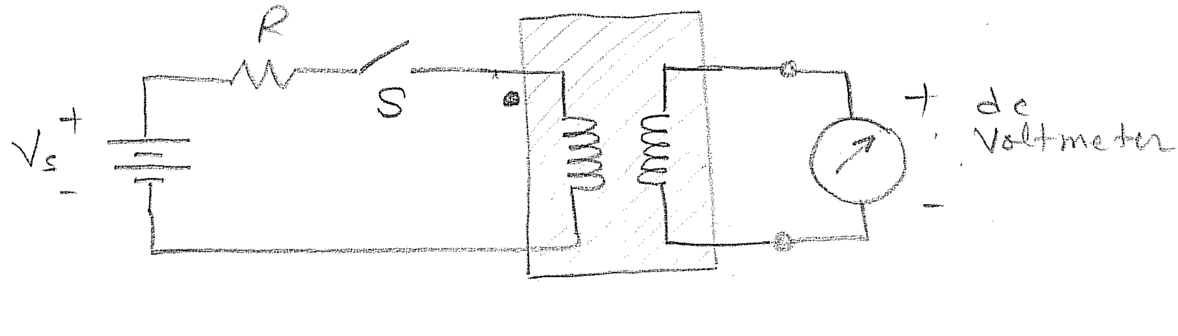
iv) Pick one terminal of the second coil, say A, and assign a current i_A entering into this terminal.

v) Use the right hand rule to determine the direction of the flux ϕ_A

vi) If the fluxes ϕ_D and ϕ_A are in the same direction then place the dot at terminal A where i_A enters. If the fluxes are of opposite direction place the dot at the terminal where the current leaves.

The dot markings may be determined experimentally. For example in transformers coils are wound on top of each other at it would be impossible to use the six-step method given above.

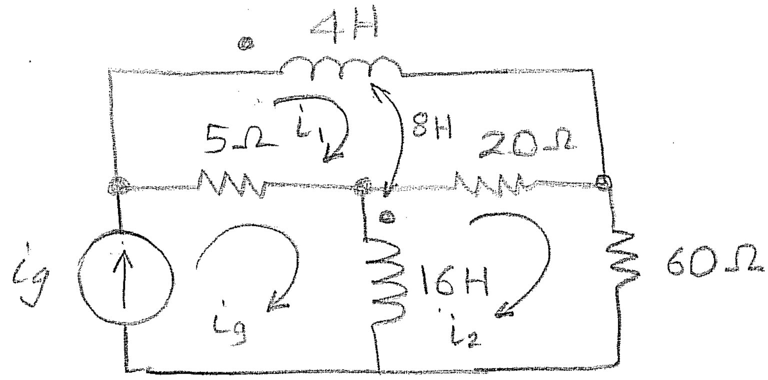
One method is to connect a dc voltage source, a resistance and switch to coil 1, The other coil (2) is connected to a voltmeter. This shown below



A dot is placed on the terminal of coil 1 connected to the positive pole of the battery. When switch S is closed, the voltmeter momentary deflection is observed. If the deflection is upscale, the terminal connected to the positive terminal of the voltmeter receives the dot mark. If the deflection is downscale, the terminal connected to the negative terminal of the voltmeter receives the dot mark.

Example 6.6: Mesh current method for a circuit with magnetically coupled coils.

a) Write a set of mesh current equations in terms of i_1 and i_2 for the circuit shown:



KVL for mesh 1:

$$4\frac{di_1}{dt} + 8\frac{d(i_g - i_2)}{dt}^{\dagger} + 20(i_1 - i_2) + 5(i_1 - i_g) = 0$$

KVL for mesh 2:

$$20(i_2 - i_1) + 60 i_2 + 16\frac{d(i_2 - i_g)}{dt} - 8\frac{di_1}{dt} = 0$$

† note that $i_g - i_2$ enters the dotted terminal so the voltage term is positive in the direction of the current (i_1).

b) Verify that if no energy is stored in the circuit at $t=0$ and if $i_g = 16 - 16e^{-5t}$, the solution for i_1 and i_2 are:

$$i_1 = 4 + 64e^{-5t} - 68e^{-4t} \quad A$$

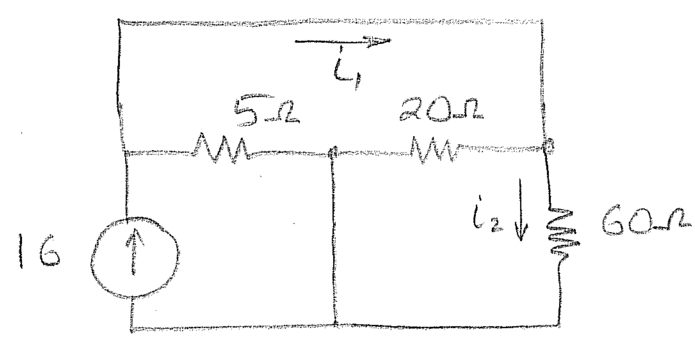
$$i_2 = 1 - 52e^{-5t} + 51e^{-4t} \quad A$$

At $t=0$

$$i_1(0) = 4 + 64 - 68 = 0$$

$$i_2(0) = 1 - 52 + 51 = 0$$

As $t \rightarrow \infty$ $i_g \rightarrow 16A$ which is a constant value, therefore the magnetically coupled coils act as a short-circuit. So the circuit reduces to:



circuit after a very long time ($t \rightarrow \infty$).

The 3 resistances are in parallel across the 16A source. $R_{eq} = (5 // 20) // 60 = 3.75\,\Omega$. So the voltage across the 16 A source is $16 \times 3.75 = 60V$.

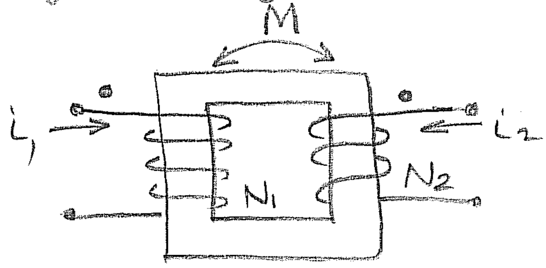
So $i_2(\infty) = \frac{60}{60} = 1A$ and $i_1(\infty) = 16 - \frac{60}{5} = 4A$

which agree with the values predicted by i_1 and i_2.

We can check that i_1 and i_2 verify the KVL equations!

Mutual Inductance in Terms of Self-Inductance

For a given magnetic circuit:



Let L_1 and L_2 be the self inductances of coils 1 and 2. It can be shown that the mutual inductance M of the two coils is given by:

$$M = k\sqrt{L_1 L_2}$$

where k is the coefficient of coupling:

$$0 \le k \le 1$$

If the coils have no common flux, then $k = 0$. When the material of the core has a high permeability most of the flux of coil 1 links coil 2. In this case $k \simeq 1$

Energy Calculations

The total energy stored in two magnetically coupled coils is given by:

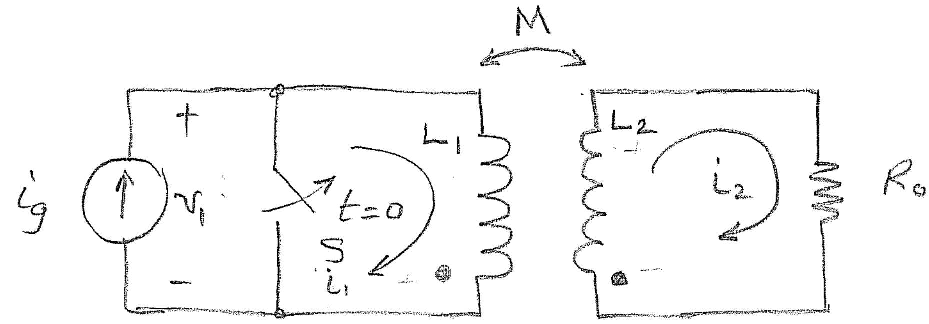
$$W = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 + i_1 i_2 M$$

The expression is valid for the currents i_1 and i_2 entering the dotted terminals.

If one of the currents leaves the dotted terminal then the term $i_1 i_2 M$ becomes $-i_1 i_2 M$!

Problem 6.39

Given the following circuit:



a) Derive the DE that governs i_2 when $L_1 = 4H$, $L_2 = 16H$, $M = 2H$, and $R_o = 32\,\Omega$.

KVL for loop 2:

$$-v_2 + i_2 R_o + L_2 \frac{di_2}{dt} = 0 \quad \text{by } v_2 = M\frac{di_1}{dt}.$$

So $\quad i_2 R_o + L_2 \frac{di_2}{dt} = M\frac{di_1}{dt}.$

$$32 i_2 + 16\frac{di_2}{dt} = 2\frac{di_1}{dt}.$$

b) When $i_1 = 8 - 8e^{-t}$ for $t \geq 0$, then verify that $i_2 = e^{-t} - e^{-2t}$ satisfies the above given DE.

Just replace and verify!

c) Find an expression for v_1:

$$v_1 = L_1\frac{di_1}{dt} - M\frac{di_2}{dt}$$

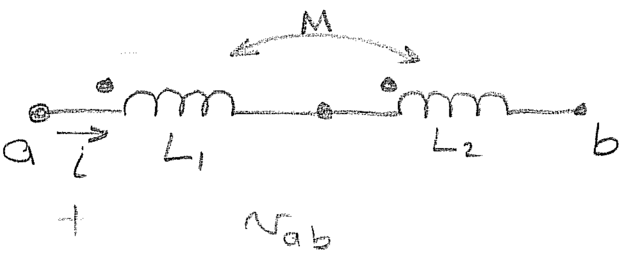
$$v_1 = 4\,(8e^{-t}) - 2\,(-e^{-t} + 2e^{-2t})$$
$$= 34e^{-t} - 4e^{-2t} \;\; V.$$

d) $v_1\,(t=0) = 30\,V.$

Problem 6.40

a) Show that the two coupled coils can be replaced by a single coil having an inductance

$L_{ab} = L_1 + L_2 + 2M$



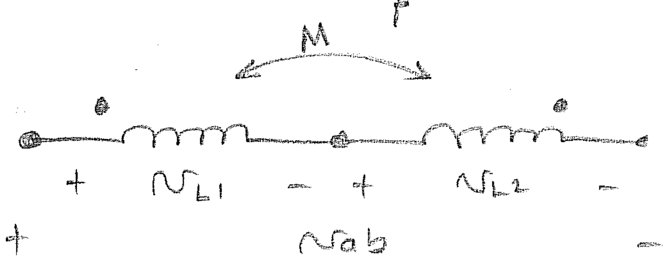
The voltage $v_{ab} = v_{L1} + v_{L2}$

$$v_{L1} = L_1 \frac{di}{dt} + M\frac{di}{dt} \qquad (1)$$

and $$v_{L2} = L_2 \frac{di}{dt} + M\frac{di}{dt}. \qquad (2)$$

So $$v_{ab} = (L_1 + L_2 + 2M)\frac{di}{dt} = L_{eq}\frac{di}{dt}.$$

b) if the connections of L_2 are reversed, i.e.:



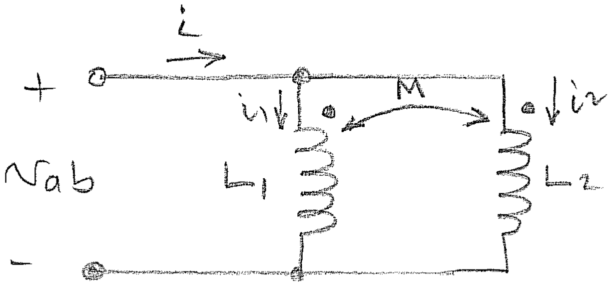
$$v_{L1} = L_1\frac{di}{dt} - M\frac{di}{dt} \quad \text{and} \quad v_{L2} = L_2\frac{di}{dt} - M\frac{di}{dt}$$

So $v_{ab} = (L_1 + L_2 - 2M)\dfrac{di}{dt}$

$= L_{eq}\dfrac{di}{dt}$.

6.41

Find the equivalent L_{eq} of the following two magnetically coupled coils



$$v_{ab} = L_1\frac{di_1}{dt} + M\frac{di_2}{dt} \quad \text{for coil 1.} \qquad (1)$$

$$v_{ab} = L_2\frac{di_2}{dt} + M\frac{di_1}{dt} \quad \text{for coil 2} \qquad (2)$$

But $i = i_1 + i_2$ (3)

Using (3) equations (1) and (2) can be written as:

$$v_{ab} = L_1\frac{di_1}{dt} + M\frac{di}{dt} - M\frac{di_1}{dt} \Rightarrow$$

$$\frac{v_{ab}}{L_1 - M} = \frac{di_1}{dt} + \frac{M}{L_1 - M}\frac{di}{dt} \qquad (4)$$

Similarly for equation (2):

$$\frac{v_{ab}}{L_2 - M} = \frac{di_2}{dt} + \frac{M}{L_2 - M}\frac{di}{dt} \qquad (5)$$

Add equations (4) and (5) noting that

$$\frac{di_1}{dt} + \frac{di_2}{dt} = \frac{di}{dt}$$ which comes from (3) !

So we obtain:

$$v_{ab}\left(\frac{1}{L_1 - M} + \frac{1}{L_2 - M}\right) = \left(1 + \frac{M}{L_1 - M} + \frac{M}{L_2 - M}\right)\frac{di}{dt}$$

Simplify to obtain:

$$v_{ab} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}\frac{di}{dt} = L_{eq}\frac{di}{dt}$$